



A large, abstract, swirling pattern in shades of blue and white occupies the upper two-thirds of the cover. The pattern is highly textured and organic, resembling turbulent fluid flow or a complex cloud formation. It has a dark blue background with lighter, more luminous, and textured areas where the swirls are more pronounced.

# Turbulent Flows

Stephen B. Pope

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1988 Cambridge University Press

ISBN 0 521 33912 0  
Library of Congress Catalog Card Number 87-21200  
British Library Cataloguing-in-Publication Data  
Pope, Stephen B.  
Turbulent flows. Cambridge, Mass.: Cambridge University Press,  
1988. viii, 500 p. £35.00  
1. Turbulence. I. Title.

[1988] Cambridge University Press  
Cambridge, New York, Port Chester, Melbourne, Sydney

© 1988 Cambridge University Press  
Printed in Great Britain by Clarendon Press, Oxford

April 8, 1988, received in library

Dk. 551.511.13  
551.511.6  
551.511.61

DIII 275

339/4122 INSTITUT  
FÜR METEOROLOGIE U. KLIMATOLOGIE  
UNIVERSITÄT HANNOVER  
HERRENHÄUSER STR. 2 - 30419 HANNOVER



CAMBRIDGE  
UNIVERSITY PRESS

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