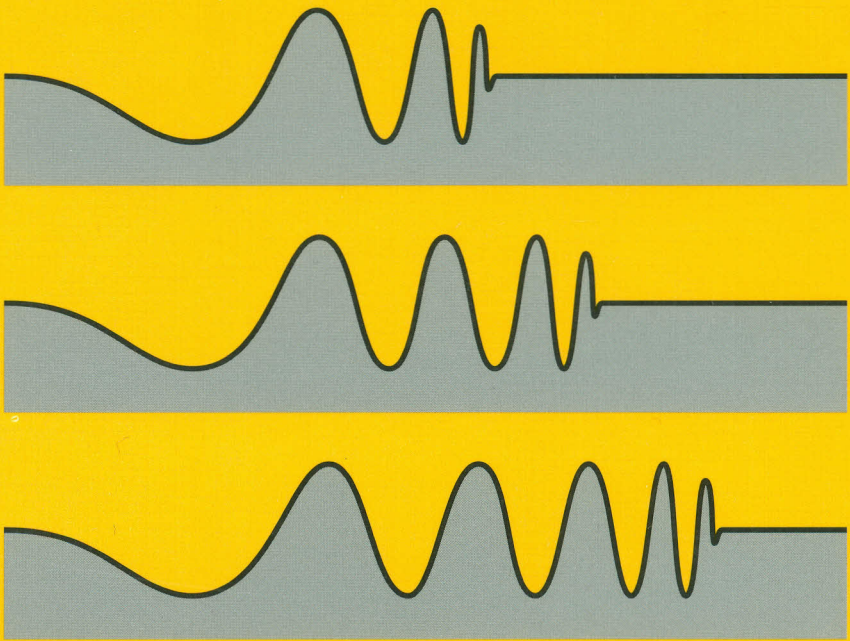


Dale R. Durran

Numerical Methods
for Wave Equations
in Geophysical
Fluid Dynamics



Springer

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Numerical Methods for Wave Equations in Geophysical Fluid Dynamics

With 93 Illustrations

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