

# Lecture Notes in Physics

Edited by H. Araki, Kyoto, J. Ehlers, München, K. Hepp, Zürich  
R. Kippenhahn, München, H. A. Weidenmüller, Heidelberg  
and J. Zittartz, Köln

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Robert Wells

Mathematical Structure  
of the Singularities  
at the Transitions  
Between Steady States  
in Hydrodynamic Systems



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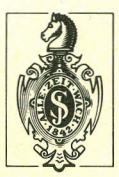
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